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EXTENDED LEVEQUE SOLUTIONS FOR FLOWS OF POWER LAW FLUIDS IN PIPES AND CHANNELS

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Abstract—A laminar flow with fully-developed velocity profile is assumed to exist in a circular pipe or rectangular channel with constant wall temperature. The Lévêque solution is the zero-order term in a power series expansion of temperature for the high Graetz number (thermal entry) flow of a Newtonian fluid. The Lévêque solution is extended here in the sense that analytical expressions for the zero-, first- and second-order terms in the power series expansion of temperature are derived for the flow of a power law fluid. The effect of heat generation by viscous dissipation is included.

NOMENCLATURE

Br,	Brinkman number;
c,	specific heat capacity;
c_1, c_2,	arbitrary constants;
e,	exponential function;
Gz,	Graetz number;
Gz^*,	modified Graetz number;
h,	channel half-width;
H,	mean heat-transfer coefficient;
k,	thermal conductivity;
L,	pipe or channel length;
m,	viscosity shear-rate exponent;
$M(,,)$,	Kummer function;
Nu,	Nusselt number;
P,	pressure drop;
Pe,	Péclet number;
Q,	volumetric flow rate;
r,	radial coordinate in pipe;
R,	pipe radius;
S,	$(T_i - T_w)/\|T_i - T_w\|$;
T,	temperature;
T_i,	inlet fluid temperature;
T_w,	wall temperature;
u,	axial velocity;
$U(,,)$,	Kummer function;
w,	channel width;
x,	axial coordinate in channel;
y,	cross-channel coordinate;
z,	axial coordinate in pipe.

Greek symbols

α, β, γ, δ, ε,	coefficients in expressions for θ_1 and θ_2;
$\Gamma(\,)$,	gamma function;
ζ,	dimensionless axial coordinate in pipe;
η,	dimensionless cross-channel coordinate;
θ,	dimensionless temperature;
θ_i,	i-th order term in power series expansion of θ;
θ_i',	first derivative of θ_i with respect to ϕ;
θ_i'',	second derivative of θ_i with respect to ϕ;
μ,	viscosity;
μ^*,	unit shear-rate viscosity;
v_1, v_2,	components of θ_2;
ξ,	dimensionless axial coordinate in channel;
ρ,	density;
σ,	dimensionless radial coordinate in pipe;
ϕ,	transformed dimensionless cross-duct coordinate in pipe or channel;
χ,	dummy variable;
ψ,	transformed dimensionless axial coordinate in pipe or channel.

1. INTRODUCTION

THERE are many practical situations in which a flow, though fully-developed from the point of view of the velocity field, is undeveloped from the point of view of the temperature field: for example, a molten polymer flow generally has a fully-developed velocity field because its viscosity is high and an undeveloped temperature field because its thermal conductivity is low. Analysis of such a flow in the simplest possible case comprises the Graetz–Nusselt problem, which is to determine the developing temperature field in a laminar flow of an incompressible Newtonian fluid with fully-developed velocity profile in a duct (generally, a circular pipe or rectangular channel). The inlet fluid temperature is different from the constant duct wall temperature, and heat generation by viscous dissipation is ignored. All fluid properties are assumed to be constant.

The Graetz–Nusselt problem may be generalized (to describe a molten polymer flow more accurately, for example) by (i) considering a power law (shear-rate dependent viscosity) fluid, of which the Newtonian fluid is a special type, and (ii) accounting for heat generation by viscous dissipation.

Analytical solutions of this (what will henceforth be called) generalized Graetz–Nusselt problem can be obtained relatively easily for flows in ducts of simple geometries (see, for example, Bird [1] and Toor [2] for flows in circular pipes). The analytical solution for the temperature field is obtained by the method of separation of variables and takes the form of an infinite sum of eigenfunctions. For long ducts, that is for low Graetz number flows, the infinite sum converges rapidly, so that accurate estimates of the temperature field can be readily obtained. For short ducts, that is for high Graetz number flows, however, the infinite sum does not converge at all rapidly and, in order to obtain accurate estimates of the temperature field, an alternative form of the solution must be obtained. It is clear that, for high Graetz number flows, the temperature field will vary significantly from the inlet fluid temperature only in a region close to the duct wall and that a boundary layer analysis is, therefore, appropriate. Such an analysis yields, as a first approximation, the analytical Lévêque solution (see, for example, Lévêque [3] and Bird, Armstrong and Hassager [4] for Lévêque solutions of the Graetz–Nusselt and generalized Graetz–Nusselt problems, respectively, for flows in circular pipes). The Lévêque solution of the Graetz–Nusselt problem has been extended analytically by Newman [5] up to and including second-order terms in a power series expansion of temperature in inverse one-third powers of local Graetz number for flows in circular pipes. The Lévêque solution is the zero-order term in this power series expansion. Mercer [6] has obtained analytically/numerically the analogous solution up to and including third-order terms for flows in rectangular channels (approximated by flows between parallel plates). The Lévêque solution of the generalized Graetz–Nusselt problem has been extended numerically by Shih and Tsou [7] up to and including fourth-order terms for flows in circular pipes. Here, the Lévêque solution of the generalized Graetz–Nusselt problem is extended analytically up to and including second-order terms for flows both in circular pipes and in rectangular channels (the latter being approximated by flows between parallel plates). The analysis is described in detail for flows in circular pipes in Section 2. The very similar analysis for flows in rectangular channels is described somewhat more briefly in Section 3. Both analyses are discussed in Section 4.

2. FLOW IN A PIPE

Let r denote the radial coordinate and z the axial coordinate in a circular pipe of radius R and length L. Let u denote the axial velocity, Q the axial volumetric flow rate, P the pressure drop, μ the viscosity, μ^* the unit shear-rate viscosity and m the viscosity shear-rate exponent (such that $\mu \equiv \mu^*|\mathrm{d}u/\mathrm{d}r|^{1/m-1}$; $m = 1$ corresponds to a Newtonian fluid). Assume that the ratio $L/2R$ is so large easy to show that

$$u = \frac{R^{m+1} - r^{m+1}}{(m+1)}\left(\frac{P}{2\mu^* L}\right)^m \tag{1}$$

and hence that

$$Q = \frac{\pi R^{m+3}}{(m+3)}\left(\frac{P}{2\mu^* L}\right)^m \tag{2}$$

(see, for example, Middleman [8]). Then, if ρ denotes the fluid density, c its specific heat capacity, k its thermal conductivity and T its temperature, the energy equation is

$$\rho c u \frac{\partial T}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \mu\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 \tag{3}$$

assuming that the Péclet number defined by

$$Pe \equiv 2\rho c Q/\pi k R \tag{4}$$

is so large that axial conduction can be neglected. Let T_i denote the inlet fluid temperature and T_w denote the constant pipe wall temperature. Then the boundary conditions on T are

$$\left.\begin{aligned} T &= T_i \quad \text{at } z = 0 \\ \frac{\partial T}{\partial r} &= 0 \quad \text{at } r = 0 \\ T &= T_w \quad \text{at } r = R. \end{aligned}\right\} \tag{5}$$

Define the Graetz number (which may be interpreted as a ratio of heat convected along the flow to heat conducted across it)

$$Gz \equiv \frac{2R}{L} Pe = \frac{4\rho c Q}{\pi k L} \tag{6}$$

and the Brinkman number (which may be interpreted as a ratio of heat generated by viscous dissipation to heat conducted between the inlet fluid flow and the pipe wall)

$$Br \equiv \frac{\mu^*(Q/2\pi R^3)^{1/m-1}Q^2}{\pi^2 R^4 k(T_i - T_w)}\frac{[2(m+3)]^{1/m+1}}{4}. \tag{7}$$

Define dimensionless variables

$$\sigma \equiv r/R,\ \zeta \equiv z/L,\ \theta \equiv (T - T_w)/(T_i - T_w) \tag{8}$$

and a modified Graetz number

$$Gz^* \equiv Gz(m+3)/4(m+1). \tag{9}$$

Then (3) and (5) become

$$Gz^*(1 - \sigma^{m+1})\frac{\partial \theta}{\partial \zeta} = \frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma\frac{\partial \theta}{\partial \sigma}\right) + Br\,\sigma^{m+1} \tag{10}$$

and

$$\left.\begin{aligned} \theta &= 1 \quad \text{at } \zeta = 0 \\ \frac{\partial \theta}{\partial \sigma} &= 0 \quad \text{at } \sigma = 0 \\ \theta &= 0 \quad \text{at } \sigma = 1 \end{aligned}\right\} \tag{11}$$

respectively. Solutions of (10) and (11) are sought which are valid for large Gz (and hence Gz^*). Significant variations in θ are then confined to a region close to the pipe wall ($\sigma = 1$) and the problem

Motivated by the form of the Lévêque solution, transform the independent variables ζ and σ by putting

$$\phi \equiv (1-\sigma)\left(\frac{2Gz^*}{9\zeta}\right)^{1/3}, \psi \equiv \left(\frac{9\zeta}{2Gz^*}\right)^{1/3}. \quad (12)$$

Then (10) and (11) become

$$[1-(1-\phi\psi)^{m+1}](1-\phi\psi)\frac{3}{2\psi^3}\left(-\phi\frac{\partial\theta}{\partial\phi}+\psi\frac{\partial\theta}{\partial\psi}\right) = -\frac{1}{\psi}\frac{\partial\theta}{\partial\phi}+\frac{(1-\phi\psi)}{\psi^2}\frac{\partial^2\theta}{\partial\phi^2}+Br(1-\phi\psi)^{m+2} \quad (13)$$

and

$$\left.\begin{array}{ll}\theta = 0 & \text{at } \phi = 0\\ \theta = 1 & \text{as } \phi \to \infty\end{array}\right\} \quad (14)$$

respectively. The boundary condition $\partial\theta/\partial\sigma = 0$ at $\sigma = 0$ is irrelevant in a boundary layer approximation.

Expand θ in a power series in ψ

$$\theta(\phi,\psi) = \theta_0(\phi)+\psi\theta_1(\phi)+\psi^2\theta_2(\phi)+\ldots. \quad (15)$$

Clearly, (15) only converges for $\psi < 1$, that is for $Gz > 18(m+1)/(m+3)$. Note that $\theta_0(\phi)$ is precisely the dimensionless form of the Lévêque solution for θ (see, for example, Bird, Armstrong and Hassager [4]) and is valid for small ψ and hence large Gz. Substitute (15) into (13) and (14), and equate coefficients of like powers of ψ to obtain

$$\theta_0''+3\left(\frac{m+1}{2}\right)\phi^2\theta_0' = 0 \quad (16)$$

$$\theta_1''+3\left(\frac{m+1}{2}\right)\phi^2\theta_1'-3\left(\frac{m+1}{2}\right)\phi\theta_1 = \left[1+\frac{3m}{2}\left(\frac{m+1}{2}\right)\phi^3\right]\theta_0' \quad (17)$$

$$\theta_2''+3\left(\frac{m+1}{2}\right)\phi^2\theta_2'-6\left(\frac{m+1}{2}\right)\phi\theta_2 = \left[1+\frac{3m}{2}\left(\frac{m+1}{2}\right)\phi^3\right]\theta_1'-\frac{3m}{2}\left(\frac{m+1}{2}\right)\phi^2\theta_1 +\left[\phi-\frac{m}{2}(m-1)\left(\frac{m+1}{2}\right)\phi^4\right]\theta_0'-Br \quad (18)$$

and so on (where $'$ denotes $d/d\phi$ and $''$ denotes $d^2/d\phi^2$) and

$$\left.\begin{array}{l}\theta_0(0) = \theta_1(0) = \theta_2(0) = \ldots = 0\\ \theta_0(\infty) = 1, \theta_1(\infty) = \theta_2(\infty) = \ldots = 0.\end{array}\right\} \quad (19)$$

Note that θ_0 and θ_1 are independent of Br. Thus, for low enough ψ, that is for high enough Gz, heat generation by viscous dissipation is unimportant (unless, of course, $T_i \equiv T_w$).

Zero-order solution

Integration of (16) is quite straightforward and yields, on substitution of the boundary conditions (19)

$$\theta_0 = \int_0^{\left(\frac{m+1}{2}\right)^{1/3}\phi} e^{-\chi^3}d\chi \Big/ \Gamma(4/3) \quad (20)$$

where $\Gamma(\,)$ denotes the gamma function (see Davis [9]).

First-order solution

Assume, by analogy with the Newtonian ($m = 1$) result of Newman [5], that

$$\theta_1 = -\frac{\alpha\phi^2}{\Gamma(4/3)}e^{-\left(\frac{m+1}{2}\right)\phi^3} - \frac{\beta\phi}{\Gamma(4/3)}\int_{\left(\frac{m+1}{2}\right)^{1/3}\phi}^{\infty} e^{-\chi^3}d\chi \quad (21)$$

where α and β are constants yet to be determined. The boundary conditions (19) require only that α and β are finite. Substitute (21) into (17), and equate coefficients of like powers of ϕ to yield

$$\left.\begin{array}{l}\alpha = \dfrac{m}{10}\left(\dfrac{m+1}{2}\right)^{1/3}\\ \beta = \dfrac{1}{2}+\dfrac{m}{10}.\end{array}\right. \quad (22)$$

Second-order solution

Let

$$\theta_2 = v_1+Br\,v_2 \quad (23)$$

v_1 satisfies (18) with $\theta_2 = v_1$ and $Br = 0$. Assume, by analogy with the Newtonian ($m = 1$) result of Newman [5], that

$$v_1 = -\frac{\alpha\phi^2}{\Gamma(4/3)}\int_{\left(\frac{m+1}{2}\right)^{1/3}\phi}^{\infty} e^{-\chi^3}d\chi - \frac{(\beta+\gamma\phi^3+\delta\phi^6)}{\Gamma(4/3)}e^{-\left(\frac{m+1}{2}\right)\phi^3} + \frac{\varepsilon}{\Gamma(4/3)}\int_0^1 \frac{\chi^{1/3}}{(1-\chi)^{2/3}}e^{-\left[\left(\frac{m+1}{2}\right)\phi^3/(1-\chi)\right]}d\chi \quad (24)$$

where α to ε are constants yet to be determined. It is easy to show that

$$\int_0^1 \frac{\chi^{1/3}}{(1-\chi)^{2/3}}e^{-\left[\left(\frac{m+1}{2}\right)\phi^3/(1-\chi)\right]}d\chi = \Gamma\left(\frac{4}{3}\right)e^{-\left(\frac{m+1}{2}\right)\phi^3}U\left(\frac{4}{3},\frac{2}{3},\left(\frac{m+1}{2}\right)\phi^3\right) \quad (25)$$

where $U(,,)$ is a Kummer function (see Slater [10]). The boundary conditions (19) require that α to ε are finite and that

$$\varepsilon = \beta\Gamma\left(\frac{5}{3}\right)\Big/3\left[\Gamma\left(\frac{4}{3}\right)\right]^2. \quad (26a)$$

Substitute (24) into (18) using (25), and equate coefficients of like powers of ϕ to yield

$$\left.\begin{aligned} \alpha &= \frac{1}{4}+\frac{m}{20} \\ \beta &= \frac{1}{840}\left(\frac{m+1}{2}\right)^{-2/3}(35+10m-m^2) \\ \gamma &= \frac{m}{420}\left(\frac{m+1}{2}\right)^{1/3}(31-m) \\ \delta &= \frac{3m^2}{200}\left(\frac{m+1}{2}\right)^{4/3} \end{aligned}\right\} \quad (26b)$$

v_2 satisfies (18) with $\theta_2 = v_2$, $\theta_0 = \theta_1 = 0$ and $Br = 1$. The solution of the homogeneous ordinary differential equation (18) with $\theta_2 = v_2$ and $\theta_0 = \theta_1 = Br = 0$ is

$$v_2 = e^{-\left(\frac{m+1}{2}\right)\phi^3}\left[c_1 M\left(\frac{4}{3},\frac{2}{3},\left(\frac{m+1}{2}\right)\phi^3\right) + c_2 U\left(\frac{4}{3},\frac{2}{3},\left(\frac{m+1}{2}\right)\phi^3\right)\right] \quad (27)$$

(see Slater [10]) where $M(,,)$ and $U(,,)$ are Kummer functions, and c_1 and c_2 are constants. The solution of the non-homogeneous ordinary differential equation is found by the method of variation of parameters (see Jeffreys and Jeffreys [11]). Incorporate the boundary conditions (19) to yield finally

$$\begin{aligned} v_2 = e^{-\left(\frac{m+1}{2}\right)\phi^3} &\times\left[M\left(\frac{4}{3},\frac{2}{3},\left(\frac{m+1}{2}\right)\phi^3\right)\left\{\frac{\Gamma(4/3)}{3\Gamma\left(\frac{5}{3}\right)\left(\frac{m+1}{2}\right)^{2/3}}\right.\right. \\ &\left. - \frac{\Gamma(4/3)\phi}{3\Gamma\left(\frac{5}{3}\right)\left(\frac{m+1}{2}\right)^{1/3}} U\left(\frac{1}{3},\frac{2}{3},\left(\frac{m+1}{2}\right)\phi^3\right)\right\} \\ &+ U\left(\frac{4}{3},\frac{2}{3},\left(\frac{m+1}{2}\right)\phi^3\right)\left\{\frac{-1}{9\left(\frac{m+1}{2}\right)^{2/3}}\right. \\ &\left.\left. + \frac{2\Gamma(4/3)\phi}{9\Gamma\left(\frac{5}{3}\right)\left(\frac{m+1}{2}\right)^{1/3}} M\left(\frac{1}{3},\frac{2}{3},\left(\frac{m+1}{2}\right)\phi^3\right)\right\}\right]. \end{aligned} \quad (28)$$

Nusselt number

Let H denote the mean heat-transfer coefficient for heat transfer from the fluid to the pipe wall based on the inlet temperature difference, so that

$$2\pi RLH|T_i - T_w| = -2\pi Rk\int_0^L \left.\frac{\partial T}{\partial r}\right|_{r=R} dz. \quad (29)$$

Define the mean Nusselt number

$$Nu \equiv 2HR/k. \quad (30)$$

Then it follows that

$$\begin{aligned} Nu = S&\left[\frac{[6(m+3)]^{1/3}}{2\Gamma(4/3)} Gz^{1/3} - \left(1+\frac{m}{5}\right)\right. \\ &- \frac{9}{560}\frac{[\Gamma(5/3)]^2}{[\Gamma(4/3)]^3}\frac{(35+10m-m^2)}{[6(m+3)]^{1/3}} Gz^{-1/3} \\ &\left. + \frac{9}{2}\frac{\Gamma(5/3)}{\Gamma(4/3)}\frac{Br}{[6(m+3)]^{1/3}} Gz^{-1/3}\right] \\ &+ O(Gz^{-2/3}) \end{aligned} \quad (31)$$

$$\left.\begin{aligned} &\text{where} \quad \Gamma(4/3) \doteqdot 0.8929795 \\ &\text{and} \quad \Gamma(5/3) \doteqdot 0.9027453 \end{aligned}\right\} \quad (32)$$

(see Davis [9]), and

$$S \equiv (T_i - T_w)/|T_i - T_w|. \quad (33)$$

Checks

The results for θ_i and θ_i' ($i = 0, 1, 2$) and Nu agree precisely with the analytical ($m = 1$, $Br = 0$) results of Newman [5]. The results for θ_i and θ_i' ($i = 0, 1, 2$) agree well with the numerical (arbitrary m and Br) results of Shih and Tsou [7] as, for example, Table 1 shows.

3. FLOW IN A CHANNEL

Let y denote the cross-channel coordinate and x the axial coordinate in a rectangular channel of half-height h, width w and length L. Assume that the ratio $w/2h$ is so large that flow in the channel can be closely approximated by flow between two parallel flat plates, and that the ratio $L/2h$ is so large that the velocity field is fully-developed. Symmetry means that attention can be confined to the region $0 \leqslant y \leqslant h$. Then it is easy to show that

$$u = \frac{h^{m+1} - y^{m+1}}{(m+1)}\left(\frac{P}{\mu^* L}\right)^m \quad (34)$$

and hence that

$$Q = \frac{2wh^{m+2}}{(m+2)}\left(\frac{P}{\mu^* L}\right)^m \quad (35)$$

(see, for example, Middleman [8]). The energy equation is

$$\rho c u\frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{du}{dy}\right)^2 \quad (36)$$

assuming that the Péclet number defined by

$$Pe \equiv \rho cQ/wk \quad (37)$$

is so large that axial conduction can be neglected. The boundary conditions on T are

$$\left.\begin{aligned} T &= T_i \quad \text{at } x = 0 \\ \frac{\partial T}{\partial y} &= 0 \quad \text{at } y = 0 \\ T &= T_w \quad \text{at } y = h. \end{aligned}\right\} \quad (38)$$

Define the Graetz number

$$Gz \equiv \frac{2h}{L} Pe = \frac{2\rho cQh}{wkL} \quad (39)$$

Table 1. Comparison of analytical (Richardson) and numerical (Shih and Tsou [7]) values of $\theta_i'(\phi = 0)$ $(i = 0, 1, 2)$ for various m and Br for flow in a pipe

m	Br	$\theta_0'(\phi=0)$ Analytical	$\theta_0'(\phi=0)$ Numerical	$\theta_1'(\phi=0)$ Analytical	$\theta_1'(\phi=0)$ Numerical	$\theta_2'(\phi=0)$ Analytical	$\theta_2'(\phi=0)$ Numerical
	0	+1.017448	+1.01745	−0.550000	−0.55000	−0.089413	−0.08941
$\frac{1}{2}$	1	+1.017448	+1.01745	−0.550000	−0.55000	+0.466926	+0.45
	5	+1.017448	+1.01745	−0.550000	−0.55000	+2.692285	+2.68571
	0	+1.119847	+1.11985	−0.600000	−0.60000	−0.089923	−0.08992
1	1	+1.119847	+1.11985	−0.600000	−0.60000	+0.415545	+0.40
	5	+1.119847	+1.11985	−0.600000	−0.60000	+2.437418	+2.41671
	0	+1.281904	+1.28190	−0.700000	−0.70000	−0.091052	−0.09105
	1	+1.281904	+1.28190	−0.700000	−0.70000	+0.350515	+0.35
2	5	+1.281904	+1.28190	−0.700000	−0.70000	+2.116783	+2.10471

and the Brinkman number

$$Br \equiv \frac{\mu^*(Q/4wh^2)^{1/m-1}Q^2}{4w^2h^2k(T_i-T_w)}\frac{[2(m+2)]^{1/m+1}}{4}. \quad (40)$$

Define dimensionless variables

$$\eta \equiv y/h,\ \xi \equiv x/L,\ \theta \equiv (T-T_w)/(T_i-T_w) \quad (41)$$

and a modified Graetz number

$$Gz^* \equiv Gz(m+2)/4(m+1). \quad (42)$$

Then (36) and (38) become

$$Gz^*(1-\eta^{m+1})\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\eta^2} + Br\,\eta^{m+1} \quad (43)$$

and

$$\left.\begin{aligned} \theta &= 1 \quad \text{at } \xi = 0 \\ \frac{\partial\theta}{\partial\eta} &= 0 \quad \text{at } \eta = 0 \\ \theta &= 0 \quad \text{at } \eta = 1 \end{aligned}\right\} \quad (44)$$

respectively. Put

$$\phi \equiv (1-\eta)\left(\frac{2Gz^*}{9\xi}\right)^{1/3},\ \psi \equiv \left(\frac{9\xi}{2Gz^*}\right)^{1/3}. \quad (45)$$

Then (43) and (44) become

$$[1-(1-\phi\psi)^{m+1}](1-\phi\psi)\frac{3}{2\psi^3}\left(-\phi\frac{\partial\theta}{\partial\phi}+\psi\frac{\partial\theta}{\partial\psi}\right)$$
$$= \frac{(1-\phi\psi)}{\psi^2}\frac{\partial^2\theta}{\partial\phi^2} + Br(1-\phi\psi)^{m+2} \quad (46)$$

[which is the same as (13) apart from one term on the RHS] and

$$\left.\begin{aligned} \theta &= 0 \quad \text{at } \phi = 0 \\ \theta &= 1 \quad \text{as } \phi \to \infty. \end{aligned}\right\} \quad (47)$$

Expand θ as in (15), which converges only for $\psi < 1$, that is for $Gz > 18(m+1)/(m+2)$, to obtain

$$\theta_0'' + 3\left(\frac{m+1}{2}\right)\phi^2\theta_0' = 0 \quad (48)$$

$$\theta_1'' + 3\left(\frac{m+1}{2}\right)\phi^2\theta_1' - 3\left(\frac{m+1}{2}\right)\phi\theta_1 = \frac{3m}{2}\left(\frac{m+1}{2}\right)\phi^3\theta_0' \quad (49)$$

$$\theta_2'' + 3\left(\frac{m+1}{2}\right)\phi^2\theta_2' - 6\left(\frac{m+1}{2}\right)\phi\theta_2 = \frac{3m}{2}\left(\frac{m+1}{2}\right)\phi^3\theta_1' - \frac{3m}{2}\left(\frac{m+1}{2}\right)\phi^2\theta_1 - \frac{m}{2}(m-1)\left(\frac{m+1}{2}\right)\phi^4\theta_0' - Br \quad (50)$$

and so on. The boundary conditions on $\theta_i (i = 0, 1, 2, \ldots)$ are (19). Note that, as for flow in a pipe, θ_0 and θ_1 are independent of Br. Thus, for high enough Gz, heat generation by viscous dissipation is unimportant (unless, of course, $T_i \equiv T_w$).

Zero-order solution

Note that (48) is identical to (16), and the boundary conditions are (19) in both cases. Hence the solution for θ_0 is given by (20).

First-order solution

Note that the RHS of (49) differs only slightly from that of (17). So assume that (21) holds, whence

$$\left.\begin{aligned} \alpha &= \frac{m}{10}\left(\frac{m+1}{2}\right)^{1/3} \\ \beta &= \frac{m}{10}. \end{aligned}\right\} \quad (51)$$

Second-order solution

Expand θ_2 as in (23). Note that the RHS of the ordinary differential equation in v_1 for a channel differs only slightly from that in v_1 for a pipe. So assume that (24) holds, whence

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= \frac{1}{840}\left(\frac{m+1}{2}\right)^{-2/3}(10m-m^2) \\ \gamma &= \frac{m}{420}\left(\frac{m+1}{2}\right)^{1/3}(10-m) \\ \delta &= \frac{3m^2}{200}\left(\frac{m+1}{2}\right)^{4/3} \end{aligned}\right\} \quad (52)$$

ε is given by (26a).

Note that v_2 for a channel satisfies the same ordinary differential equation and boundary conditions as v_2 for a pipe. Hence the solution for v_2 is given by (28).

Nusselt number

Let H denote the mean heat-transfer coefficient for heat transfer from the fluid to the channel wall based on the inlet temperature difference, so that

$$2wLH|T_i - T_w| = -2wk \int_0^L \left.\frac{\partial T}{\partial y}\right|_{y=h} \mathrm{d}x. \quad (53)$$

Define the mean Nusselt number

$$Nu \equiv 2Hh/k. \quad (54)$$

Then it follows that

$$Nu = S\left[\frac{[6(m+2)]^{1/3}}{2\Gamma(4/3)} Gz^{1/3} - \left(\frac{m}{5}\right) - \frac{9}{560}\frac{[\Gamma(5/3)]^2}{[\Gamma(4/3)]^3}\frac{(10m-m^2)}{[6(m+2)]^{1/3}} Gz^{-1/3} + \frac{9}{2}\frac{\Gamma(5/3)}{\Gamma(4/3)}\frac{Br}{[6(m+2)]^{1/3}} Gz^{-1/3}\right] + O(Gz^{-2/3}) \quad (55)$$

where $\Gamma(4/3)$, $\Gamma(5/3)$ and S are given by (32) and (33).

Checks

The results for θ_i and $\theta_i'(i = 0, 1, 2)$ agree well with the analytical/numerical ($m = 1$, $Br = 0$) results of Mercer [6] as, for example, Table 2 shows.

Table 2. Comparison of analytical (Richardson) and analytical/numerical (Mercer [6]) values of $\theta_i'(\phi = 0)$ $(i = 0, 1, 2)$ with $m = 1$ and $Br = 0$ for flow in a channel

	$\theta_0'(\phi = 0)$	$\theta_1'(\phi = 0)$	$\theta_2'(\phi = 0)$
Analytical	+1.119847	−0.100000	−0.018393
Analytical/ numerical	+1.12250	−0.10012	−0.01832

4. CONCLUSION

The comparisons made in Sections 2 and 3 of the results obtained here with those obtained elsewhere act as checks on the analysis described here. In order to determine the range of applicability of the results obtained here, on the other hand, comparison must be made with the results of a full solution of the generalized Graetz–Nusselt problem. Comparison with the (exact) results of Bird [1], Toor [2], Toor [12], Lyche and Bird [13] and White [14] for flows in circular pipes indicates that the relative error in the (asymptotic) results obtained here is of order 0.01 for $Gz \geqslant 1000$, of order 0.1 for $Gz \geqslant 100$ and of order 1 for $Gz \geqslant 10$ as, for example, Table 3 shows. (It should, however, be noted when making the comparison that the results of a full solution of the generalized Graetz–Nusselt problem are themselves liable to error for large Gz; the reason for performing the analysis described here is, after all, precisely that its results can be obtained both more accurately and more quickly than the results of a full solution of the generalized Graetz–Nusselt problem for large Gz.) The extended Lévêque solutions described here may, therefore, be regarded as supplements to the full eigenfunction sum solutions of the generalized Graetz–Nusselt problem, the former being valid for $Gz \geqslant 1000$ and the latter valid (or, more strictly, useful) for $Gz \leqslant 1000$.

Table 3. Comparison of asymptotic (Richardson) and exact (White [14]) analytical values of Nu with $m = 1$ and $Br = 0$ for flow in a pipe

Gz	Nu Asymptotic	Exact
10	2.149	4.156
100	6.236	7.155
1000	14.923	15.384

It was noted at the appropriate points in sections 2 and 3 that θ_0 (and θ_1) is independent of Br and hence that, for large enough Gz, heat generation by viscous dissipation is unimportant (unless, of course, $T_i \equiv T_w$). This result can be made more precise by a comparison of the expressions for θ_0 and θ_2 which indicates that heat generation by viscous dissipation is unimportant and may be neglected unless Br is at least comparable with $[(m+3)Gz]^{2/3}$ for flows in circular pipes and with $[(m+2)Gz]^{2/3}$ for flows in rectangular channels (assuming, of course, that $Gz \geqslant 1000$; otherwise, no such statement can be made).

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SOLUTIONS DE LEVEQUE ETENDUES AUX ECOULEMENTS DE FLUIDES EN LOI PUISSANCE DANS LES TUBES ET LES CANAUX

Résumé—On admet qu'un écoulement laminaire, à profil de vitesse établi, existe dans un tube circulaire ou un canal rectangulaire avec une température de paroi constante. La solution de Lévêque est le terme d'ordre zéro dans une série puissance de la température pour les grands nombres de Graetz (entrée thermique) et un fluide newtonien. La solution de Lévêque est étendue de telle sorte que les expressions analytiques pour les termes d'ordre zéro, un et deux dans la série puissance de la température sont obtenues pour l'écoulement d'un fluide à loi puissance. L'effet de la génération de chaleur par dissipation visqueuse est inclus.

ERWEITERTE LÉVÊQUE-LÖSUNGEN FÜR STRÖMUNGEN VON FLUIDEN, DIE DEM POTENZGESETZ GENÜGEN, IN ROHREN UND KANÄLEN

Zusammenfassung—In einem kreisförmigen Rohr oder rechteckigen Kanal mit konstanter Wandtemperatur wird eine laminare Strömung mit vollkommen ausgebildetem Geschwindigkeitsprofil angenommen. Die Lösung nach Lévêque ist der Term nullter Ordnung in einer Potenzreihenentwicklung der Temperatur für die Strömung einer Newtonschen Flüssigkeit mit großer Graetz-Zahl (thermischer Einlauf). Die Lösung nach Lévêque wird hier in der Form erweitert, daß die analytischen Ausdrücke für die Terme nullter, erster und zweiter Ordnung in der Potenzreihenentwicklung der Temperatur für die innere Strömung eines Fluides, das dem Potenzgesetz genügt, hergeleitet werden. Der Einfluß der Wärmeerzeugung durch viskose Dissipation wird dabei berücksichtigt.

ОБОБЩЕНИЕ РЕШЕНИЯ ЛЕВЕКА НА СЛУЧАЙ ТЕЧЕНИЯ СТЕПЕННЫХ ЖИДКОСТЕЙ В ТРУБАХ И КАНАЛАХ

Аннотация — Предполагается, что в круглой трубе или прямоугольном канале с постоянной температурой стенок имеет место ламинарное течение с полностью развитым профилем скорости. Решением Левека для течения ньютоновской жидкости при больших значениях числа Гретца (тепловой начальный участок) является член нулевого порядка в степенном разложении температуры. Проведенное в данной работе обобщение решения Левека заключается в том, что аналитические выражения для членов нулевого, первого и второго порядков по числу Гретца в степенном разложении температуры найдены для потока степенной жидкости. Учитывается эффект выделения тепла за счёт вязкой диссипации.